ON THE LIFESPAN OF AND THE BLOWUP MECHANISM FOR SMOOTH SOLUTIONS TO A CLASS OF 2-D NONLINEAR WAVE EQUATIONS WITH SMALL INITIAL DATA

DING Bingbing, Ingo WITT, and YIN Huicheng

Abstract. This paper is concerned with the lifespan and the blowup mechanism for smooth solutions to the 2-D nonlinear wave equation $\partial_t^2 u - \sum_{i=1}^{2} \partial_i (c_i^2(u)\partial_i u) = 0$, where $c_i(u) \in C^\infty(\mathbb{R}^n)$, $c_i(0) \neq 0$, and $(c_1'(0))^2 + (c_2'(0))^2 \neq 0$. This equation has an interesting physics background as it arises from the pressure-gradient model in compressible fluid dynamics and also in nonlinear variational wave equations. Under the initial condition $(u(0,x), \partial_t u(0,x)) = (\varepsilon u_0(x), \varepsilon u_1(x))$ with $u_0(x), u_1(x) \in C_0^\infty(\mathbb{R}^2)$, and $\varepsilon > 0$ is small, we will show that the classical solution $u(t,x)$ stops to be smooth at some finite time T_ε. Moreover, blowup occurs due to the formation of a singularity of the first-order derivatives $\nabla_{t,x} u(t,x)$, while $u(t,x)$ itself is continuous up to the blowup time T_ε.

§1. Introduction and main result

In this paper, we are concerned with the lifespan T_ε of and the blowup mechanism for classical solutions to the 2-D nonlinear wave equation

$$\begin{cases} \partial_t^2 u - \displaystyle\sum_{i=1}^{2} \partial_i (c_i^2(u)\partial_i u) = 0, \\ (u(0,x), \partial_t u(0,x)) = (\varepsilon u_0(x), \varepsilon u_1(x)), \end{cases} \tag{1.1}$$

with small initial data, where $c_i(u) \in C^\infty(\mathbb{R}^n)$, $c_i(0) \neq 0$, and $(c_1'(0))^2 + (c_2'(0))^2 \neq 0$. In addition, $u_0(x), u_1(x) \in C_0^\infty(\mathbb{R}^2)$ and $\varepsilon > 0$ is sufficiently small.

In case $c_1(u) = c_2(u) = e^{u/2}$, Eq. (1.1) has a background in physics as it arises from the pressure-gradient model in compressible Euler systems and is rather analogous to the 2-D variational wave equation $\partial_t^2 u - \sum_{i=1}^{2} c(u)\partial_i(c(u)\partial_i u) = 0$ (for the physics backgrounds of the variational wave equation and its mathematical treatment, see [2, 10, 15, 24] and the references therein).

Here is a derivation of the pressure-gradient model with small initial data: As pointed out in [1, 30-31], the pressure-gradient system is a simplified version of the compressible Euler equations, which arises from

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splitting the compressible Euler system (i.e., the inertia terms $\operatorname{div}(\rho U)$, $\operatorname{div}(\rho U\otimes U)$ and the pressure p are considered separately). It has the form

$$\begin{cases} \partial_t\rho = 0, \\ \partial_t(\rho U) + \nabla p = 0, \\ \partial_t(\rho E) + \operatorname{div}(pU) = 0, \end{cases} \tag{1.2}$$

where ρ is density, $U = (u_1, u_2)$ is velocity, p is pressure, $E = \dfrac{1}{2}|U|^2 + \dfrac{1}{\gamma-1}\dfrac{p}{\rho}$ is energy, and γ is the adiabatic exponent with $1 < \gamma < 3$.

For simplicity, as in [19-20, 25], we assume $\rho \equiv 1$ in (1.2). In this case, (1.2) becomes

$$\begin{cases} \partial_t U + \nabla p = 0, \\ \partial_t E + \operatorname{div}(pU) = 0. \end{cases} \tag{1.3}$$

It follows from the transformation $p = (\gamma-1)P$, $t = \dfrac{T}{\gamma-1}$ and (1.3) that

$$\begin{cases} \partial_T U + \nabla P = 0, \\ \partial_T P + P\operatorname{div} U = 0. \end{cases} \tag{1.4}$$

Let us consider the following Cauchy problem for (1.4):

$$\begin{cases} \partial_T U + \nabla P = 0, \\ \partial_T P + P\operatorname{div} U = 0, \\ U|_{T=0} = \varepsilon U_0(x), \quad P|_{T=0} = 1 + \varepsilon P_0(x), \end{cases} \tag{1.5}$$

where $\varepsilon > 0$ is small and $U_0(x) = (u_1^0(x), u_2^0(x)) \in C_0^\infty(\mathbb{R}^2)$, $P_0(x) \in C_0^\infty(\mathbb{R}^2)$ are supported in the disc $B(0, M)$. One then obtains that P satisfies the nonlinear wave equation

$$\partial_T\left(\frac{\partial_T P}{P}\right) - \Delta P = 0. \tag{1.6}$$

Let $v(T, x) = \ln P$. Then it follows from (1.6) and the initial data in (1.5) that

$$\begin{cases} \partial_T^2 v - \operatorname{div}(e^v\nabla v) = 0, \\ v(0, x) = \ln(1 + \varepsilon P_0(x)) = \varepsilon P_0(x) + \varepsilon^2\displaystyle\sum_{n=2}^{\infty}(-1)^{n-1}\frac{\varepsilon^{n-2}}{n!}P_0^n(x), \\ \partial_t v(0, x) = -\varepsilon\operatorname{div} U_0(x). \end{cases} \tag{1.7}$$

In (1.7), use t and $u(t, x)$ in place of T and $v(T, x)$, respectively. As a nonlinear problem equivalent to (1.7), one can then consider

$$\begin{cases} \partial_t^2 u - \operatorname{div}(e^u\nabla u) = 0, \quad (t, x) \in [0, \infty)\times\mathbb{R}^2, \\ u(0, x) = \varepsilon u_0(x), \\ \partial_t u(0, x) = \varepsilon u_1(x), \end{cases} \tag{1.8}$$

where $u_0(x) = P_0(x)$ and $u_1(x) = -\operatorname{div} U_0(x)$. In this way we have given a brief derivation on the nonlinear wave equation in the form (1.1) from the fundamental equations of compressible fluid dynamics.

Without loss of generality, we will assume that $c_i(0) = 1$ $(i = 1, 2)$ in (1.1). Since third-order terms like $O(u^2D^2u)$ and $O(u|Du|^2)$ will not have an essential influence on the blowup behavior of small data solution to problem (1.1), Eq. (1.1) is basically equivalent to

$$\begin{cases} \partial_t^2 u - \sum_{i=1}^{2} \partial_i \left((1 + c_i u)\partial_i u\right) = 0, \quad (t, x) \in [0, \infty) \times \mathbb{R}^2, \\ (u(0, x), \partial_t u(0, x)) = (\varepsilon u_0(x), \varepsilon u_1(x)), \end{cases} \tag{1.9}$$

where $c_1 = 2c_1'(0), c_2 = 2c_2'(0)$, and $c_1^2 + c_2^2 \neq 0$.

We introduce polar coordinates (r, θ) in $\mathbb{R}^2$,

$$\begin{cases} x_1 = r\cos\theta, \\ x_2 = r\sin\theta, \end{cases}$$

where $r = \sqrt{x_1^2 + x_2^2}$, $\theta \in [0, 2\pi]$, and $\omega \equiv (\omega_1, \omega_2) = (\cos\theta, \sin\theta)$. Later we will need the function

$$F_0(\sigma, \theta) \equiv \frac{1}{2^{3/2}\pi} \int_\sigma^{+\infty} \frac{R(s, \omega; u_1) - \partial_s R(s, \omega; u_0)}{\sqrt{s - \sigma}}\, ds, \tag{1.10}$$

where $\sigma \in \mathbb{R}$, and $R(s, \omega; v)$ is the Radon transform of the smooth function $v(x)$, i.e., $R(s, \omega; v) = \int_{x\cdot\omega=s} v(x)\, dS$. ¿From Theorem 6.2.2 and (6.2.12) of [14], one has that the function $F_0(\sigma, \theta) \not\equiv 0$ unless $u_0(x) \equiv 0$ and $u_1(x) \equiv 0$. Moreover, $F_0(\sigma, \theta) \equiv 0$ for $\sigma \geq M$ and $\lim_{\sigma\to-\infty} F_0(\sigma, \theta) = 0$. Therefore,

$$\min_{\sigma,\theta}[\partial_\sigma F_0(\sigma, \theta)(c_1\cos^2\theta + c_2\sin^2\theta)] < 0$$

exists as long as $(u_0(x), u_1(x)) \not\equiv 0$.

We will assume throughout this paper that there is a unique point (σ^0, θ^0) such that

$$\begin{cases} \partial_\sigma F_0(\sigma^0, \theta^0)(c_1\cos^2\theta^0 + c_2\sin^2\theta^0) = \min_{\sigma\in\mathbb{R},\, \theta\in[0,2\pi]}[\partial_\sigma F_0(\sigma, \theta)(c_1\cos^2\theta + c_2\sin^2\theta)], \\ \text{the Hessian matrix } \nabla^2_{\sigma,\theta}[\partial_\sigma F_0(\sigma, \theta)(c_1\cos^2\theta + c_2\sin^2\theta)]|_{(\sigma,\theta)=(\sigma^0,\theta^0)} > 0. \end{cases} \tag{1.11}$$

Let T_ε denote the lifespan of the smooth solution to (1.9). Then one has:

Theorem 1.1. *Let $u_0(x)$, $u_1(x) \in C_0^\infty(\mathbb{R}^2)$ be supported in the disc $B(0, M)$ and let assumption* (1.11) *hold. Then*:

(1)

$$\lim_{\varepsilon\to0} \varepsilon\sqrt{T_\varepsilon} = \tau_0 \equiv -\frac{1}{\partial_\sigma F_0(\sigma^0, \theta^0)(c_1\cos^2\theta^0 + c_2\sin^2\theta^0)} > 0. \tag{1.12}$$

(2) *There exists a point $M_\varepsilon = (T_\varepsilon, x_\varepsilon)$ and a positive constant C independent of ε such that*
- (i) *$u(t, x) \in C([0, T_\varepsilon] \times \mathbb{R}^2)$ and $\|u\|_{L^\infty([0,T_\varepsilon]\times\mathbb{R}^2)} \leq C\varepsilon$.*
- (ii) *$u \in C^2([0, T_\varepsilon] \times \mathbb{R}^2 \setminus \{M_\varepsilon\})$, and, for $t < T_\varepsilon$, it satisfies*

$$\|\nabla_{t,x} u(t, \cdot)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C}{T_\varepsilon - t}, \tag{1.13}$$

$$\|\partial_t u(t, \cdot)\|_{L^\infty(\mathbb{R}^2)} \geq \frac{1}{C(T_\varepsilon - t)}. \tag{1.14}$$

Remark 1.1. Compared with the "lifespan theorems" of [4-5], Theorem 1.1 states that the solution $u(t,x)$ to (1.9) is continuous up to the blowup time $t = T_\varepsilon$, while its first-order derivatives $\nabla_{t,x}u$ develop a singularity at $t = T_\varepsilon$. In the terminology of [4-5], this corresponds to an "ODE blowup." On the contrary, the blowup result of [4-5] on small data solutions to the 2-D nonlinear wave equation $\partial_t^2 v - \Delta_x v + \sum_{0\le i,j,k\le 2} g_{ij}^k \partial_k v\, \partial_{ij}^2 v = 0$, where the nonlinearity depends on the derivatives of v, but not v itself, shows that the solution $v(t,x)$ is C^1 up to the blowup time T_ε, while the second-order derivatives $\nabla_{t,x}^2 v$ develop a singularity at $t = T_\varepsilon$. In the terminology of [4-5], this is a "geometric blowup."

Remark 1.2. One readily obtains $u(t,x) \in C^\infty(([0,T_\varepsilon]\times\mathbb{R}^2)\setminus\{M_\varepsilon\})$ from $u(t,x) \in C^2(([0,T_\varepsilon]\times\mathbb{R}^2)\setminus\{M_\varepsilon\})$ in Theorem 1.1. Since $u(t,x) \in C^\infty([0,T_\varepsilon)\times\mathbb{R}^2)$, this in fact follows from the property of finite propagation speed which holds for hyperbolic equations.

Remark 1.3. In particular, for the 2-D pressure-gradient model $\partial_t^2 u - \sum_{i=1}^{2} \partial_i(e^u \partial_i u) = 0$ with small initial data $(u(0,x), \partial_t u(0,x)) = (\varepsilon u_0(x), \varepsilon u_1(x))$ and $(u_0(x), u_1(x)) \not\equiv 0$, it follows from Theorem 1.1 that the lifespan T_ε of the smooth solution $u(t,x)$ satisfies $\lim_{\varepsilon\to 0} \varepsilon\sqrt{T_\varepsilon} = -\frac{1}{\min_{\sigma\in\mathbb{R},\,\theta\in[0,2\pi]} \partial_\sigma F_0(\sigma,\theta)}$ under an assumption on the function $\partial_\sigma F_0(\sigma,\theta)$ that is analogous to (1.11). We thus have extended the blowup result of [21] valid for the rotationally symmetric case to this now more general situation. In addition, returning to the original pressure-gradient system (1.5), one obtains that $\partial_t P$ and $\operatorname{div} U$ develop a singularity at time $t = T_\varepsilon$. This corresponds to the formation of a shock emanating from the blowup point as shown in [26] for the compressible Euler system.

Remark 1.4. The nonlinear equation (1.9) can be rewritten as $\partial_t^2 u - (1+u)\Delta u = |\nabla u|^2$ when $c_1 = c_2 = 1$. For the 3-D equation $\partial_t^2 u - (1+u)\Delta u = 0$ with small initial data $(u(0,x), \partial_t u(0,x)) = (\varepsilon u_0(x), \varepsilon u_1(x))$, in [6, 23] it was shown that smooth solutions exist globally. On the other hand, for the n-dimensional nonlinear wave equation ($n = 2,3$) with coefficients depending on the derivatives of the solution, $\partial_t^2 u - c^2(\partial_t u)\Delta u = 0$ and, more generally, $\sum_{i,j=0}^{n} g_{ij}(\nabla u)\partial_{ij}^2 u = H(\nabla u)$, where $t = x_0$, $x = (x_1, ..., x_n)$, $g_{ij}(\nabla u) = c_{ij} + O(|\nabla u|)$, $H(\nabla u) = O(|\nabla u|^2)$, and the linear part $\sum_{i,j=0}^{n} c_{ij}\partial_{ij}^2 u$ is strictly hyperbolic with respect to time t, it is known that small data smooth solutions exist globally if related null conditions hold (see [8, 14] and others), while otherwise small data smooth solutions blow up in finite time (see [4-5, 13, 17, 22] and others). We point out that in the case considered here the coefficients of the nonlinear equation (1.9) depend on both the solution u and its derivatives.

Near the blowup point M_ε one can give a more accurate description of the behavior of the solution $u(t,x)$ which is similar to statements in the "geometric blowup theorems" of [4–5].

Theorem 1.2. *Assume that the constants τ_1, A_0, A_1 and δ_0 satisfy $0 < \tau_1 < \tau_0$, $A_0 < \sigma^0 < A_1 < M$ and that $\delta_0 > 0$ is sufficiently small. Moreover, assume that A_0 and A_1 are close to σ^0. Denote by D the domain*

$$D \equiv \{(s,\theta,\tau) \mid A_0 \le s \le A_1,\ \theta^0 - \delta_0 \le \theta \le \theta^0 + \delta_0,\ \tau_1 \le \tau \le \tau_\varepsilon\},$$

where $\tau_\varepsilon = \varepsilon\sqrt{T_\varepsilon}$. Then there exist a subdomain D_0 of D containing a point $m_\varepsilon = (s_\varepsilon, \theta_\varepsilon, \tau_\varepsilon)$ and functions $\phi(s,\theta,\tau)$, $v(s,\theta,\tau) \in C^3(D_0)$ with the following properties:

(1) *In the domain D_0, ϕ satisfies*

$$\begin{cases} \partial_s\phi(s,\theta,\tau) \ge 0, \quad \partial_s\phi(s,\theta,\tau) = 0 \Longleftrightarrow (s,\theta,\tau) = m_\varepsilon, \\ \partial_{\tau s}^2\phi(m_\varepsilon) < 0, \quad \nabla_{s,\theta}\partial_s\phi(m_\varepsilon) = 0, \quad \nabla_{s,\theta}^2\partial_s\phi(m_\varepsilon) > 0. \end{cases} \tag{H}$$

(2) *It holds*

$$v(m_\varepsilon) \neq 0. \tag{1.15}$$

Moreover, let the function $G(\sigma, \theta, \tau)$ be defined by $G(\Phi) = v(s, \theta, \tau)$ in the domain $\Phi(D_0)$, where Φ is a map such that $\Phi(s, \theta, \tau) = (\phi(s, \theta, \tau), \theta, \tau)$. Then $u(t, x) = \frac{\varepsilon}{\sqrt{r}} G(r - t, \theta, \varepsilon\sqrt{t})$ solves Eq. (1.9).

Remark 1.5. Theorem 1.2 provides a more accurate description of the solution near the blowup point $M_\varepsilon = \Phi(m_\varepsilon)$ than Theorem 1.1. First, one has that $G(\sigma, \theta, \tau) \in C(\Phi(D_0))$ because of $\phi, v \in C^3(D_0)$ and (H) of Theorem 1.2. To prove this assertion, we are only required to show that G is continuous at the point $M_\varepsilon = (\sigma_\varepsilon, \theta_\varepsilon, \tau_\varepsilon) \equiv (\phi(m_\varepsilon), \theta_\varepsilon, \tau_\varepsilon)$. To this end, let $(\sigma_n, \theta_n, \tau_n) \in \Phi(D_0)$ be such that $(\sigma_n, \theta_n, \tau_n) \to (\sigma_\varepsilon, \theta_\varepsilon, \tau_\varepsilon)$ as $n \to \infty$. It then follows from (H) that there is a unique point $(s_n, \theta_n, \tau_n) \in D_0$ such that $\sigma_n = \phi(s_n, \theta_n, \tau_n)$. By Taylor's formula, one has $\sigma_n - \sigma_\varepsilon = \nabla_{\theta,\tau}\phi(m_\varepsilon) \cdot (\theta_n - \theta_\varepsilon, \tau_n - \tau_\varepsilon) + \frac{1}{2}\partial^2_{s\tau}\phi(m_\varepsilon)(s_n - s_\varepsilon)(\tau_n - \tau_\varepsilon) + \frac{1}{2}(\theta_n - \theta_\varepsilon, \tau_n - \tau_\varepsilon)\nabla^2_{\theta,\tau}\phi(m_\varepsilon)(\theta_n - \theta_\varepsilon, \tau_n - \tau_\varepsilon)^T + \frac{1}{6}\partial^3_s\phi(m_\varepsilon)(s_n - s_\varepsilon)^3 + o(|s_n - s_\varepsilon|^3) + o(|\theta_n - \theta_\varepsilon| + |\tau_n - \tau_\varepsilon|)$. Together with $\partial^3_s\phi(m_\varepsilon) > 0$, this yields $s_n \to s_\varepsilon$ as $n \to \infty$. Therefore, one obtains $G \in C(\Phi(D_0))$ from $G(\Phi) = v$ and the continuity of v, ϕ in D_0. It follows that $u(t, x) = \frac{\varepsilon}{\sqrt{r}} G(r - t, \theta, \varepsilon\sqrt{t}) \in C\left(\left[\frac{\tau_1^2}{\varepsilon^2}, T_\varepsilon\right] \times \mathbb{R}^2\right) \cap C^1\left(\left(\left[\frac{\tau_1^2}{\varepsilon^2}, T_\varepsilon\right] \times \mathbb{R}^2\right) \setminus \{M_\varepsilon\}\right)$ and $\|u\|_{L^\infty} \leq C\varepsilon^2$ in a neighborhood of M_ε. Regarding the other properties of $u(t, x)$ near M_ε stated in Theorem 1.1, see §4 below for details.

There are some interesting papers on the Riemann problem for the pressure-gradient system (1.5) and (1.6), respectively, with special discontinuous initial data, with either a mathematical treatment or a numerical simulation (see [1, 19-20, 25, 29-31] and the references therein). There are also many results on the blowup of classical solutions and the global existence and uniqueness of weak solutions, respectively, to 1-D variational wave equations (see [2, 7, 10-12, 16, 27-28] and the references therein). In the multidimensional case of Eq. (1.1), however, except for the rotationally symmetric case, where in [9, 21] blowup results have been established, until now there were no results on the finite-time blowup of smooth solutions to (1.1) or even on mechanisms of this blowup. In this paper, we shall focus on these two problems, i.e., we will establish the precise lifespan T_ε in Theorem 1.1 and determine the blowup mechanism in Theorem 1.2.

Let us comment on the proofs of Theorems 1.1 and 1.2. First we derive the required lower bound on the lifespan T_ε for solutions to problem (1.9). As in [14, Chapter 6] and [13], by constructing a suitable approximate solution $u_a(t, x)$ to (1.9) and then considering the difference of the exact solution $u(t, x)$ and $u_a(t, x)$, applying the Klainerman-Sobolev inequality, and further establishing a delicate energy estimate, we obtain this lower bound on the lifespan T_ε. Next we derive the required upper bound on T_ε. Motivated by the "geometric blowup" method of [4-5], we introduce the blowup system of (1.9) to study simultaneously the lifespan T_ε and blowup mechanism of smooth solution u. That is, by introducing a singular change of coordinates Φ in the domain $D = \{(\sigma, \theta, \tau) \mid -C_0 \leq \sigma \leq M,\ 0 \leq \theta \leq 2\pi,\ 0 < \tau_1 \leq \tau \leq \tau_\varepsilon\}$,

$$(s, \theta, \tau) \to (\phi(s, \theta, \tau), \theta, \tau), \text{ where } \phi(s, \theta, \tau_1) = s \text{ and } \partial_s\phi = 0 \text{ holds at some point,}$$

where $\sigma = r - t$, $\tau = \varepsilon\sqrt{t}$, and $C_0 > 0$ a fixed constant, and setting $G(\Phi) = v(s, \theta, \tau)$, we obtain a nonlinear system for (ϕ, v) from the ansatz $u(t, x) = \frac{\varepsilon}{\sqrt{r}} G(r - t, \theta, \varepsilon\sqrt{t})$ and the equation in (1.9). This blowup system for (1.9) has a unique smooth solution (ϕ, v) for $\tau \leq \tau_\varepsilon$, where the couple (ϕ, v) satisfies properties (H) and (1.15) of Theorem 1.2. This enables us to determine the blowup point at time $t = T_\varepsilon$ for the solution u of (1.9) and give a complete asymptotic expansion of T_ε as well as a precise description of the behavior of $u(t, x)$ close to the blowup point. In order to treat the resulting blowup system, as in [4-5], we use the Nash-Moser-Hörmander iteration method to overcome the difficulties introduced by the free boundary $t = T_\varepsilon$ and the inherent complexity of the nonlinear blowup system. To this end, the linearized system is solved first. Thanks to the energy estimates established in [4-5], we are then able to complete the proof of Theorem 1.2.

The paper is organized as follows: In §2, as in [9, 21], we construct a suitable approximate solution $u_a(t,x)$ to (1.9) and establish related estimates, which allows us to obtain the required lower bound on the lifespan T_ε. In §3, the blowup system for (1.9) is solved, which allows us to prove Theorem 1.2. Then, in §4, we conclude the proof of Theorem 1.1 based on Theorem 1.2.

Notation. Throughout the paper, we will use the following notation: Z denotes one of the Klainerman vector fields in $\mathbb{R}_t^+ \times \mathbb{R}^2$, i.e.,

$$\partial_t,\ \ \partial_i,\ \ \Gamma_0 = t\partial_t + \sum_{j=1}^{2} x_j\partial_j,\ \ H_i = x_i\partial_t + t\partial_i,\ \ i=1,2,\ \ R = x_1\partial_2 - x_2\partial_1,$$

∂ stands for ∂_t or ∂_i $(i=1,2)$, and ∇_x stands for (∂_1,∂_2).

§2. Lower bound on the lifespan T_ε

In this section, we establish the lower bound of T_ε for smooth solution to the Cauchy problem (1.9).

Let $\tau = \varepsilon\sqrt{1+t}$ be the slow time variable and assume the solution to (1.9) can be approximated by

$$\frac{\varepsilon}{\sqrt{r}} V(\sigma,\theta,\tau), \qquad r>0,$$

where $\sigma = r-t$, $(x_1,x_2) = (r\cos\theta, r\sin\theta)$ with $\theta\in[0,2\pi]$.

The function $V(\sigma,\theta,\tau)$ solves the equation

$$\begin{cases} \partial^2_{\sigma\tau}V + (c_1\cos^2\theta + c_2\sin^2\theta)V\partial_\sigma^2 V + (c_1\cos^2\theta + c_2\sin^2\theta)(\partial_\sigma V)^2 = 0, \\ V(\sigma,\theta,0) = F_0(\sigma,\theta), \\ \operatorname{supp} V(\cdot,\theta,\tau) \subseteq \{\sigma\le M\}, \end{cases} \tag{2.1}$$

where $F_0(\sigma,\theta)$ has been defined in (1.10).

For problem (2.1), one has:

Lemma 2.1. Eq. (2.1) *admits a C^∞ solution for $0\le\tau<\tau_0$ with the number τ_0 being given in* (1.12).

Proof. Set $U(\sigma,\theta,\tau) = \partial_\sigma V(\sigma,\theta,\tau)$. Then it follows from (2.1) that

$$\begin{cases} \partial_\tau U + (c_1\cos^2\theta + c_2\sin^2\theta)V\partial_\sigma U + (c_1\cos^2\theta + c_2\sin^2\theta)U^2 = 0, \\ U(\sigma,\theta,0) = \partial_\sigma F_0(\sigma,\theta). \end{cases} \tag{2.2}$$

The characteristic curve $\sigma = \sigma(s,\theta,\tau)$ of (2.2) starting at the point $(s,\theta,0)$ is defined by

$$\begin{cases} \dfrac{d\sigma}{d\tau}(s,\theta,\tau) = (c_1\cos^2\theta + c_2\sin^2\theta)V(\sigma(s,\theta,\tau),\theta,\tau), \\ \sigma(s,\theta,0) = s. \end{cases} \tag{2.3}$$

Along this characteristic curve, it follows from (2.2) that, for $\tau<\tau_0$,

$$U(\sigma(s,\theta,\tau),\theta,\tau) = \frac{\partial_\sigma F_0(s,\theta)}{1+(c_1\cos^2\theta + c_2\sin^2\theta)\partial_\sigma F_0(s,\theta)\tau}. \tag{2.4}$$

Because of $U(\sigma,\theta,\tau) = \partial_\sigma V(\sigma(s,\theta,\tau),\theta,\tau)$, from (2.3)-(2.4) one then obtains that

$$\begin{cases} \partial^2_{\tau s}\sigma(s,\theta,\tau) = \dfrac{(c_1\cos^2\theta + c_2\sin^2\theta)\partial_\sigma F_0(s,\theta)}{1+(c_1\cos^2\theta + c_2\sin^2\theta)\partial_\sigma F_0(s,\theta)\tau}\,\partial_s\sigma(s,\theta,\tau), \\ \partial_s\sigma(s,\theta,0) = 1. \end{cases}$$

This gives $\partial_s\sigma(s,\theta,\tau) = 1 + (c_1\cos^2\theta + c_2\sin^2\theta)\partial_\sigma F_0(s,\theta)\tau > 0$ for $0 \le \tau < \tau_0$ and then

$$\sigma(s,\theta,\tau) = \sigma(M,\theta,\tau) + s - M + (c_1\cos^2\theta + c_2\sin^2\theta)F_0(s,\theta)\tau \tag{2.5}$$

and

$$V(\sigma(s,\theta,\tau),\theta,\tau) = \frac{\partial_\tau\sigma(M,\theta,\tau)}{c_1\cos^2\theta + c_2\sin^2\theta} + F_0(s,\theta). \tag{2.6}$$

Note that $\sigma(M,\theta,\tau) = M$ such that $V(\sigma,\theta,\tau)$ satisfies the boundary condition $V|_{\sigma=M} = 0$. This, together with (2.5)-(2.6), yields $V(\sigma,\theta,\tau) = F_0(s,\theta)$ and $\sigma = s + (c_1\cos^2\theta + c_2\sin^2\theta)F_0(s,\theta)\tau$. By the implicit function theorem, one then has that $s = s(\sigma,\theta,\tau)$ is a smooth function of σ,θ,τ for $\tau < \tau_0$. Therefore, $V(\sigma,\theta,\tau) = F_0(s(\sigma,\theta,\tau),\theta)$ is a smooth solution of (2.1) for $0 \le \tau < \tau_0$ as claimed. □

¿From [14, Chapter 6], one has that $F_0(\sigma,\theta) \in C^\infty(\mathbb{R})$ is supported in $(-\infty, M]$ and obeys the estimates

$$\left|\partial_\sigma^k\partial_\theta^l F_0(\sigma,\theta)\right| \le C_{kl}\,(1+|\sigma|)^{-1/2-k}, \quad k \in \mathbb{N}_0. \tag{2.7}$$

¿From (2.7), we now derive a decay estimate of $V(\sigma,\theta,\tau)$ in (2.1) for $\tau < \tau_0$ and $\sigma \to -\infty$.

Lemma 2.2. *For any positive constant $b < \tau_0$, one has that, in the domain*

$$\{(\sigma,\theta,\tau) \mid -\infty < \sigma \le M, 0 \le \theta \le 2\pi, 0 \le \tau \le b\},$$

and for $r \ge t/3$, the smooth solution V to (2.1) *obeys the estimates*

$$|Z^\alpha\partial_\tau^l\partial_\sigma^m V(\sigma,\theta,\tau)| \le C_{\alpha b}^{lm}(1+|\sigma|)^{-1/2-l-m}, \quad \alpha, l, m \in \mathbb{N}_0, \tag{2.8}$$

where $C_{\alpha b}^{lm}$ are positive constants depending on b and α, l, m.

Proof. When $\tau \le b$, it follows from (2.5) and the support property of $F_0(\sigma,\theta)$ that $\frac{|s|}{2} \le |\sigma| \le 2|s|$ for large $|s|$. Together with (2.6), this yields

$$|V(\sigma,\theta,\tau)| \le C_b\,(1+|\sigma|)^{-1/2}, \quad |\partial_\sigma V(\sigma,\theta,\tau)| \le C_b\,(1+|\sigma|)^{-3/2}. \tag{2.9}$$

By (2.6) and (2.4), one has

$$\partial_\sigma s(\sigma,\theta,\tau) = \frac{1}{1 + \left(c_1\cos^2\theta + c_2\sin^2\theta\right)\partial_s F_0(s,\theta)\tau}$$

and

$$\partial_\sigma^2 V(\sigma(s,\theta,\tau),\theta,\tau) = \frac{\partial_s^2 F_0(s,\theta)}{(1 + \left(c_1\cos^2\theta + c_2\sin^2\theta\right)\partial_s F_0(s,\theta)\tau)^3},$$

which yields

$$|\partial_\sigma^2 V(\sigma,\theta,\tau)| \le C_b\,(1+|\sigma|)^{-5/2}. \tag{2.10}$$

Further, it follows from (2.1) and (2.10) that

$$|\partial_{\tau\sigma}^2 V(\sigma,\theta,\tau)| \le C_b\,(1+|\sigma|)^{-3}$$

and then

$$|\partial_\tau V(\sigma,\theta,\tau)| \le C_b\,(1+|\sigma|)^{-2}. \tag{2.11}$$

Based on (2.9)–(2.11), by an inductive argument one arrives at

$$|\partial_\tau^l \partial_\sigma^m V(\sigma,\theta,\tau)| \le C_b^{lm}\,(1+|\sigma|)^{-1/2-l-m}, \quad l,m\in\mathbb{N}_0.$$

Because of

$$\Gamma_0=\sigma\partial_\sigma+\frac{\varepsilon t}{2\sqrt{1+t}}\,\partial_\tau, \quad H_1=-\sigma\cos\theta\partial_\sigma+\frac{\varepsilon x_1}{2\sqrt{1+t}}\,\partial_\tau-\frac{x_2 t}{r^2}\,\partial_\theta,$$
$$H_2=-\sigma\sin\theta\partial_\sigma+\frac{\varepsilon x_2}{2\sqrt{1+t}}\,\partial_\tau+\frac{x_1 t}{r^2}\,\partial_\theta, \quad R=\partial_\theta,$$

one analogously obtains

$$|Z^\alpha\partial_\tau^l \partial_\sigma^m V(\sigma,\theta,\tau)| \le C_{\alpha b}^{lm}\,(1+|\sigma|)^{-1/2-l-m}, \quad \alpha,l,m\in\mathbb{N}_0,$$

which completes the proof of Lemma 2.2. $\square$

Next, we construct an approximate solution u_a to (1.9) for $0\le\tau=\varepsilon\sqrt{1+t}<\tau_0$.
Let w_0 be the solution of the linear wave equation

$$\begin{cases} \partial_t^2 w_0-\triangle w_0=0, \\ w_0(0,x)=u_0(x), \quad \partial_t w_0(0,x)=u_1(x). \end{cases}$$

It follows from [14, Theorem 6.2.1] that, for any constants $l>0$ and $0<m<1$,

$$\left|Z^\alpha(w_0(t,x)-r^{-1/2}F_0(\sigma,\theta))\right| \le C_{\alpha l}\,(1+t)^{-3/2}(1+|\sigma|)^{1/2}, \quad r\ge lt, \tag{2.12}$$
$$\left|\partial^k w_0(t,x)\right| \le C_{km}\,(1+t)^{-1-|k|}, \quad r\le mt. \tag{2.13}$$

Choose a C^∞ function $\chi(s)$ such that $\chi(s)=1$ for $s\le 1$ and $\chi(s)=0$ for $s\ge 2$. For $0\le\tau=\varepsilon\sqrt{1+t}<\tau_0$, we take the approximate solution u_a to (1.9) to be

$$u_a(t,x)=\varepsilon\left(\chi(\varepsilon t)w_0(t,x)+r^{-1/2}(1-\chi(\varepsilon t))\chi(-3\varepsilon\sigma)V(\sigma,\tau)\right). \tag{2.14}$$

By Lemma 2.2 and [14, Theorem 6.2.1], one has that, for a fixed positive constant $b<\tau_0$,

$$|Z^\alpha u_a(t,x)| \le C_{\alpha b}\,\varepsilon(1+t)^{-1/2}(1+|\sigma|)^{-1/2}, \quad \tau\le b. \tag{2.15}$$

Set $J_a=\partial_t^2 u_a-(1+c_1u_a)\partial_1^2 u_a-(1+c_2u_a)\partial_2^2u_a-c_1(\partial_1u_a)^2-c_2(\partial_2u_a)^2$.

Lemma 2.3. *One has*

$$\int_0^{b^2/\varepsilon^2-1}\|Z^\alpha J_a(t,\cdot)\|_{L^2}\,dt \le C_{\alpha b}\,\varepsilon^{3/2}.$$

Proof. We divide the proof into three parts.

(A) $0\le t\le \dfrac{1}{\varepsilon}$**.** In this case, $\chi(\varepsilon t)=1$ and $u_a=\varepsilon w_0$. This yields

$$J_a=-\varepsilon^2 w_0c_1\partial_1^2w_0-\varepsilon^2w_0c_2\partial_2^2w_0-\varepsilon^2c_1(\partial_1w_0)^2-\varepsilon^2c_2(\partial_2w_0)^2.$$

It follows from (2.15) and a direct computation that, for $0\le t\le\dfrac{1}{\varepsilon}$,

$$\|Z^\alpha J_a(t,\cdot)\|_{L^2}\le C\,\varepsilon^2(1+t)^{-1/2}. \tag{2.16}$$

(B) $\frac{1}{\varepsilon} \le t \le \frac{2}{\varepsilon}$**.** We now rewrite u_a as

$$u_a = \varepsilon w_0(t,x) + \varepsilon(1-\chi(\varepsilon t))\Big(r^{-1/2}\chi(-3\varepsilon\sigma)V(\sigma,\theta,\tau) - w_0(t,x)\Big).$$

Then

$$J_a = J_1 + J_2 + J_3 + J_4, \tag{2.17}$$

where

$$\begin{aligned}
J_1 &= -c_1 u_a \partial_1^2 u_a - c_2 u_a \partial_2^2 u_a - c_1(\partial_1 u_a)^2 - c_2(\partial_2 u_a)^2,\\
J_2 &= \varepsilon(\partial_t^2 - \Delta)\Big[(1-\chi(\varepsilon t))r^{-1/2}\chi(-3\varepsilon\sigma)\big(V(\sigma,\theta,\tau) - F_0(\sigma,\theta)\big)\Big],\\
J_3 &= \varepsilon(\partial_t^2 - \Delta)\Big[(1-\chi(\varepsilon t))\chi(-3\varepsilon\sigma)\big(r^{-1/2}F_0(\sigma,\theta) - w_0(t,x)\big)\Big],\\
J_4 &= \varepsilon(\partial_t^2 - \Delta)\left[(1-\chi(\varepsilon t))(\chi(-3\varepsilon\sigma)-1)w_0(t,x)\right].
\end{aligned}$$

We treat each term J_i $(1 \le i \le 4)$ in (2.17) separately. ¿From (2.15) one obtains

$$\|Z^\alpha J_1(t,\cdot)\|_{L^2} \le C_{\alpha b}\,\varepsilon^2(1+t)^{-1/2}. \tag{2.18}$$

Since

$$\begin{aligned}
J_2 = {}& \varepsilon r^{-1/2}(\partial_t - \partial_r)(\partial_t + \partial_r)\left[(1-\chi(\varepsilon t))\chi(-3\varepsilon\sigma)\big(V(\sigma,\theta,\tau) - F_0(\sigma,\theta)\big)\right]\\
&- \frac{\varepsilon}{4} r^{-5/2}(1-\chi(\varepsilon t))\chi(-3\varepsilon\sigma)\big(V(\sigma,\theta,\tau) - F_0(\sigma,\theta)\big)\\
&- \varepsilon r^{-5/2}\partial_\theta^2\left[(1-\chi(\varepsilon t))\chi(-3\varepsilon\sigma)\big(V(\sigma,\theta,\tau) - F_0(\sigma,\theta)\big)\right]
\end{aligned}$$

and $V(\sigma,\theta,\tau) - F_0(\sigma,\theta) = \int_0^\tau \partial_\tau V(\sigma,\theta,s)ds$, one has

$$\|Z^\alpha J_2(t,\cdot)\|_{L^2} \le C_{\alpha b}\,\varepsilon^2(1+t)^{-1/2}. \tag{2.19}$$

Note that $-\frac{2}{3\varepsilon} \le \sigma \le M$ holds on the support of J_3 which implies $r \ge \frac{1}{3}t$. This, together with (2.12), yields

$$\|Z^\alpha J_3(t,\cdot)\|_{L^2} \le C_\alpha\,\varepsilon^2(1+t)^{-1/2}. \tag{2.20}$$

Analogously, together with (2.13), one arrives at

$$\|Z^\alpha J_4(t,\cdot)\|_{L^2} \le C_{\alpha b}\,\varepsilon^2(1+t)^{-1}. \tag{2.21}$$

Collecting (2.18)-(2.21) yields

$$\|Z^\alpha J_a(t,\cdot)\|_{L^2} \le C_{\alpha b}\,\varepsilon^2(1+t)^{-1/2}, \quad \frac{1}{\varepsilon} \le t \le \frac{2}{\varepsilon}. \tag{2.22}$$

(C) $\frac{2}{\varepsilon} \le t \le \frac{b^2}{\varepsilon^2} - 1$**.** A direct computation yields

$$\begin{aligned}
J_a = {}& -\varepsilon^2 r^{-1/2}\partial_{\tau\sigma}^2\hat{V}\left(\frac{1}{\sqrt{1+t}} - r^{-1/2}\right)\\
&- \varepsilon^2 r^{-1}\Big(\partial_{\tau\sigma}^2\hat{V} + (c_1\cos^2\theta + c_2\sin^2\theta)\hat{V}\partial_\sigma^2\hat{V} + (c_1\cos^2\theta + c_2\sin^2\theta)(\partial_\sigma\hat{V})^2\Big)\\
&+ O(\varepsilon^2)\,(1+t)^{-3/2}(1+|\sigma|)^{-1/2},
\end{aligned} \tag{2.23}$$

where $\hat{V}(\sigma,\theta,\tau)=\chi(-3\varepsilon\sigma)V(\sigma,\theta,\tau)$. It follows from (2.1) that

$$\varepsilon^2 r^{-1}\bigg(\partial^2_{\tau\sigma}\hat{V}+(c_1\cos^2\theta+c_2\sin^2\theta)\hat{V}\partial^2_\sigma\hat{V}+(c_1\cos^2\theta+c_2\sin^2\theta)(\partial_\sigma\hat{V})^2\bigg)$$
$$=O(\varepsilon^2)\,(1+t)^{-3/2}(1+|\sigma|)^{-3/2}, \quad (2.24)$$

here we have used the fact that $\chi(-3\varepsilon\sigma)(1-\chi(-3\varepsilon\sigma)$ is supported in the interval $[-\frac{2}{3\varepsilon},-\frac{1}{3\varepsilon}]$. Substituting (2.24) into (2.23) yields

$$\|Z^\alpha J_a(t,\cdot)\|_{L^2}\le C_{\alpha b}\varepsilon^2(1+t)^{-3/4}. \quad (2.25)$$

Consequently, combining (2.16), (2.22), and (2.25) yields

$$\int_0^{b^2/\varepsilon^2-1}\|Z^\alpha J_a(t,\cdot)\|_{L^2}\,dt\le C_{\alpha b}\,\varepsilon^{3/2},$$

which completes the proof of Lemma 2.3. $\Box$

Lemma 2.4. *For sufficiently small ε and $0\le\tau=\varepsilon\sqrt{1+t}\le b<\tau_0$,* Eq. (1.9) *admits a C^∞ solution u which satisfies the estimate*

$$|Z^\kappa\partial(u-u_a)|\le C_b\,\varepsilon^{3/2}(1+t)^{-1/2}(1+|t-r|)^{-1/2}. \quad (2.26)$$

for $|\kappa|\le 2$.

Proof. Set $v=u-u_a$. Then

$$\begin{cases}\partial_t^2 v-(1+c_1u)\partial_1^2 v-(1+c_2u)\partial_2^2 v=F,\\ v(0,x)=\partial_t v(0,x)=0,\end{cases} \quad (2.27)$$

where

$$F=-J_a+c_1v\partial_1^2u_a+c_2v\partial_2^2u_a+c_1(\partial_1v)^2+c_2(\partial_2v)^2+2c_1(\partial_1v)(\partial_1u_a)+2c_2(\partial_2v)(\partial_2u_a). \quad (2.28)$$

We will use continuous induction to prove (2.26). To this end, we assume that, for some $T\le\frac{b^2}{\varepsilon^2}-1$,

$$|Z^\kappa\partial v|\le\varepsilon\,(1+t)^{-1/2}(1+|t-r|)^{-1/2},\quad |\kappa|\le 2,\ \ t\le T, \quad (2.29)$$

holds and subsequently we prove that

$$|Z^\kappa\partial v|\le\frac{\varepsilon}{2}\,(1+t)^{-1/2}(1+|t-r|)^{-1/2},\quad |\kappa|\le 2,\ \ t\le T. \quad (2.30)$$

Note that from (2.29) one has

$$|Z^\kappa v|\le C\varepsilon\,(1+t)^{-1/2}(1+|t-r|)^{1/2},\quad |\kappa|\le 2,\ \ t\le T. \quad (2.31)$$

Applying Z^α to both hand sides of (2.27) yields, for $|\alpha|\le 4$,

$$\big(\partial_t^2-(1+c_1u)\partial_1^2-(1+c_2u)\partial_2^2\big)Z^\alpha v=G$$
$$\equiv\sum_{|\beta|\le|\alpha|}C_{\alpha\beta}Z^\beta F+\big[Z^\alpha,c_1u\partial_1^2+c_2u\partial_2^2)]v\quad+\sum_{|\beta|<|\alpha|}C'_{\alpha\beta}Z^\beta\big(c_1u\partial_1^2v+c_2u\partial_2^2v\big), \quad (2.32)$$

where the commutator relation $[Z^\alpha, \partial_t^2 - \triangle] = \sum_{|\beta|<|\alpha|} C''_{\alpha\beta} Z^\beta (\partial_t^2 - \triangle)$ was used, and $C_{\alpha\beta}$, $C'_{\alpha\beta}$, $C''_{\alpha\beta}$ are suitable constants.

Next we derive from (2.32) an estimate of $\|\partial Z^\alpha v(t,\cdot)\|_{L^2}$. Define the energy

$$E(t) = \frac{1}{2} \sum_{|\alpha|\le 4} \int_{\mathbb{R}^2} (|\partial_t Z^\alpha v|^2 + (1+c_1 u)(\partial_1 Z^\alpha v)^2 + (1+c_2 u)(\partial_2 Z^\alpha v)^2)\, dx.$$

Multiplying both sides of (2.32) by $\partial_t Z^\alpha v$ ($|\alpha| \le 4$), integrating by parts in $\mathbb{R}^2$, and noting that $|\partial u| = |\partial u_a + \partial v| \le C_b \varepsilon (1+t)^{-1/2}$ from the construction of u_a and assumption (2.29), one arrives at

$$E'(t) \le \frac{C_b \varepsilon}{\sqrt{1+t}} E(t) + \sum_{|\alpha|\le 4} \int_{\mathbb{R}^2} |G| \cdot |\partial_t Z^\alpha v|\, dx. \tag{2.33}$$

Moreover, due to the inductive hypothesis (2.29) and (2.15), one has

$$|Z^\kappa u| \le C_b \varepsilon\, (1+t)^{-1/2} (1+|\sigma|)^{1/2} \le C_b \varepsilon, \quad |\kappa| \le 2, \ \ t \le T. \tag{2.34}$$

We now treat each term in the sum $\sum_{|\alpha|\le 4} \int_{\mathbb{R}^2} |G| \cdot |\partial_t Z^\alpha v|\, dx$ separately.

(A) Estimation of $\sum_{|\beta|<|\alpha|} \int_{\mathbb{R}^2} |Z^\beta \big(c_1 u \partial_1^2 v + c_2 u \partial_2^2 v\big)| \cdot |\partial_t Z^\alpha v|\, dx$**.** It follows from (2.34) that, for $|\beta| < |\alpha|$, $i = 1, 2$,

$$\begin{aligned}
\int_{\mathbb{R}^2} |Z^\beta (u \partial_i^2 v)| \cdot |\partial_t Z^\alpha v|\, dx &\le C_b \sum_{|\beta_1|+|\beta_2|=|\beta|} \int_{\mathbb{R}^2} |Z^{\beta_1} u| \cdot |Z^{\beta_2} \partial_i^2 v| \cdot |\partial_t Z^\alpha v|\, dx \\
&\le C_b \sum_{|\beta_1|+|\beta_2|=|\beta|} \int_{\mathbb{R}^2} |Z^{\beta_1} v| \cdot |Z^{\beta_2} \partial_i^2 v| \cdot |\partial_t Z^\alpha v|\, dx \\
&\quad + C_b \sum_{|\beta_1|+|\beta_2|=|\beta|} \int_{\mathbb{R}^2} |Z^{\beta_1} u_a| \cdot |Z^{\beta_2} \partial_i^2 v| \cdot |\partial_t Z^\alpha v|\, dx.
\end{aligned} \tag{2.35}$$

Due to

$$\partial_t = \frac{t\Gamma_0 - \sum_{i=1}^{2} x_i H_i}{t^2 - r^2}, \quad \partial_1 = \frac{x_2 R + t H_1 - x_1 \Gamma_0}{t^2 - r^2}, \quad \partial_2 = \frac{-x_1 R + t H_2 - x_2 \Gamma_0}{t^2 - r^2},$$

one then has

$$|Z^{\beta_2} \partial_i^2 v| \le \frac{2}{1+|t-r|} \sum_{|\beta_2'|=|\beta_2|+1} |Z^{\beta_2'} \partial v|.$$

Because of $|\beta| < |\alpha| \le 4$, (2.29), and the fact that $\big\|(1+|t-r|^{-1} f)(t,\cdot)\big\|_{L^2} \le \|\partial f(t,\cdot)\|_{L^2}$ for the function $f(t,x) \in C^1(\mathbb{R}^+ \times \mathbb{R}^2)$ with $\mathrm{supp} f \subseteq \{r \le M + t\}$ (this inequality can be found in [22]), the first term in the right-hand side of (2.35) can be estimated as

$$\begin{aligned}
\int_{\mathbb{R}^2} |Z^{\beta_1} v| \cdot |Z^{\beta_2} \partial_i^2 v| \cdot |\partial_t Z^\alpha v|\, dx &\le C_b \sum_{|\beta_2'|=|\beta_2|+1} \int_{\mathbb{R}^2} \frac{1}{1+|t-r|} |Z^{\beta_1} v| \cdot |Z^{\beta_2'} \partial v| \cdot |\partial_t Z^\alpha v|\, dx \\
&\le \frac{C_b \varepsilon}{\sqrt{1+t}} E(t).
\end{aligned} \tag{2.36}$$

Analogously,

$$\int_{\mathbb{R}^2} |Z^{\beta_1} u_a| \cdot |Z^{\beta_2} \partial_i^2 v| \cdot |\partial_t Z^\alpha v| \, dx \le \frac{C_b \varepsilon}{\sqrt{1+t}} \, E(t).$$

Therefore, one obtains

$$\sum_{|\beta|<|\alpha|} \int_{\mathbb{R}^2} |Z^\beta \big(c_1 u \partial_1^2 v + c_2 u \partial_2^2 v\big)| \cdot |\partial_t Z^\alpha v| \, dx \le \frac{C_b \varepsilon}{\sqrt{1+t}} \, E(t). \tag{2.37}$$

(B) Estimation of $\int_{\mathbb{R}^2} \big|[Z^\alpha, c_1 u \partial_1^2 + c_2 u \partial_2^2] v\big| \cdot |\partial_t Z^\alpha v| \, dx$**.** For $i = 1, 2$,

$$\begin{aligned}
&\int_{\mathbb{R}^2} \big|[Z^\alpha, u \partial_i^2] v\big| \cdot |\partial_t Z^\alpha v| \, dx \\
&\quad \le C_b \sum_{\substack{|\alpha_1|+|\alpha_2|=|\alpha| \\ |\alpha_1|\ge 1}} \int_{\mathbb{R}^2} |Z^{\alpha_1} u| \cdot |Z^{\alpha_2} \partial_i^2 v| \cdot |\partial_t Z^\alpha v| \, dx + \int_{\mathbb{R}^2} |u| \cdot \big|[Z^\alpha, \partial_i^2] v\big| \cdot |\partial_t Z^\alpha v| \, dx \\
&\quad \le C_b \left(\sum_{\substack{|\alpha_1|+|\alpha_2|=|\alpha| \\ |\alpha_1|\ge 1}} \int_{\mathbb{R}^2} |Z^{\alpha_1} u_a| \cdot |Z^{\alpha_2} \partial_i^2 v| \cdot |\partial_t Z^\alpha v| \, dx + \sum_{|\beta|<|\alpha|} \int_{\mathbb{R}^2} |u| \cdot |\partial^2 Z^\beta v| \cdot |\partial_t Z^\alpha v| \, dx \right. \\
&\qquad \left. + \sum_{\substack{|\alpha_1|+|\alpha_2|=|\alpha| \\ |\alpha_1|\ge 1}} \int_{\mathbb{R}^2} |Z^{\alpha_1} v| \cdot |Z^{\alpha_2} \partial_i^2 v| \cdot |\partial_t Z^\alpha v| \, dx \right).
\end{aligned}$$

By the same argument as in (2.37), one then has

$$\int_{\mathbb{R}^2} \big|[Z^\alpha, c_1 u \partial_1^2 + c_2 u \partial_2^2] v\big| \cdot |\partial_t Z^\alpha v| \, dx \le \frac{C_b \varepsilon}{\sqrt{1+t}} \, E(t). \tag{2.38}$$

Next we treat each of the terms $\int_{\mathbb{R}^2} |Z^\beta F| \cdot |\partial_t Z^\alpha v| \, dx$, $|\beta| \le |\alpha|$, which are included in $\sum_{|\alpha|\le 4} \int_{\mathbb{R}^2} |G| \cdot |\partial_t Z^\alpha v| \, dx$.

(C) Estimation of $\int_{\mathbb{R}^2} |Z^\beta J_a| \cdot |\partial_t Z^\alpha v| \, dx$**.** In this case, one has

$$\int_{\mathbb{R}^2} |Z^\beta J_a| \cdot |\partial_t Z^\alpha v| \, dx \le \|Z^\beta J_a\|_{L^2} \sqrt{E(t)}. \tag{2.39}$$

(D) Estimation of $\int_{\mathbb{R}^2} |Z^\beta \big(c_1 v \partial_1^2 u_a + c_2 v \partial_2^2 u_a\big)| \cdot |\partial_t Z^\alpha v| \, dx$**.** Due to (2.34) a direct computation yields, for $i = 1, 2$,

$$\begin{aligned}
\int_{\mathbb{R}^2} |Z^\beta \big(v \partial_i^2 u_a\big)| \cdot |\partial_t Z^\alpha v| \, dx &\le C_b \sum_{|\beta_1|+|\beta_2|=|\beta|} \int_{\mathbb{R}^2} |Z^{\beta_1} v| \cdot |Z^{\beta_2} \partial_i^2 u_a| \cdot |\partial_t Z^\alpha v| \, dx \\
&\le C_b \sum_{\substack{|\beta_1|+|\beta_2|=|\beta| \\ |\beta_2'|=|\beta_2|+1}} \int_{\mathbb{R}^2} \frac{1}{1+|t-r|} |Z^{\beta_1} v| \cdot |Z^{\beta_2'} \partial u_a| \cdot |\partial_t Z^\alpha v| \, dx \\
&\le \frac{C_b \varepsilon}{\sqrt{1+t}} \, E(t).
\end{aligned} \tag{2.40}$$

(E) Estimation of $\int_{\mathbb{R}^2} |Z^\beta\big(c_1(\partial_1 v)^2 + c_2(\partial_2 v)^2\big)| \cdot |\partial_t Z^\alpha v|\, dx$. Similar to (D), one has

$$\int_{\mathbb{R}^2} |Z^\beta\big(c_1(\partial_1 v)^2 + c_2(\partial_2 v)^2\big)| \cdot |\partial_t Z^\alpha v| dx \le \frac{C_b\varepsilon}{\sqrt{1+t}}\, E(t). \tag{2.41}$$

(F) Estimation of $\int_{\mathbb{R}^2} |Z^\beta\big(c_1(\partial_1 v)(\partial_1 u_a) + c_2(\partial_2 v)(\partial_2 u_a)\big)| \cdot |\partial_t Z^\alpha v|\, dx$. It follows by direct computation that, for $i = 1, 2$,

$$\begin{aligned}\int_{\mathbb{R}^2} |Z^\beta\big((\partial_i v)(\partial_i u_a)\big)| \cdot |\partial_t Z^\alpha v|\, dx &\le C_b \sum_{|\beta_1|+|\beta_2|\le|\beta|} \int_{\mathbb{R}^2} |Z^{\beta_1}\partial v| \cdot |Z^{\beta_2}\partial u_a| \cdot |\partial_t Z^\alpha v|\, dx \\ &\le \frac{C_b\varepsilon}{\sqrt{1+t}}\, E(t). \end{aligned}\tag{2.42}$$

Substituting (2.37)-(2.42) into (2.33) yields

$$E'(t) \le \frac{C_b\varepsilon}{\sqrt{1+t}}\, E(t) + \sum_{|\beta|\le 4} \|Z^\beta J_a(t,\cdot)\|_{L^2} \sqrt{E(t)}.$$

Thus, by Lemma 2.3 and Gronwall's inequality, one obtains

$$\|\partial Z^\alpha v(t,\cdot)\|_{L^2} \le C_b\, \varepsilon^{3/2}, \quad |\alpha| \le 4,$$

and further

$$\|Z^\alpha \partial v(t,\cdot)\|_{L^2} \le C_b\, \varepsilon^{3/2}, \quad |\alpha| \le 4. \tag{2.43}$$

By (2.43) and the Klainerman-Sobolev inequality (see [14,18]), one has

$$|Z^\kappa \partial v| \le C_b\, \varepsilon^{3/2}(1+t)^{-1/2}(1+|t-r|)^{-1/2}, \quad |\kappa| \le 2,\ \ t \le T, \tag{2.44}$$

which means that, for small ε,

$$|Z^\kappa \partial v| \le \frac{\varepsilon}{2}\,(1+t)^{-1/2}(1+|t-r|)^{-1/2}, \quad |\kappa| \le 2,\ \ t \le T.$$

This completes the proofs of (2.29) and (2.26). $\square$

Proof of the lower bound on T_ε. Lemma 2.4 implies that $\varliminf_{\varepsilon\to 0} \varepsilon\sqrt{1+T_\varepsilon} \ge \tau_0$ holds for the lifespan T_ε of smooth solutions to (1.9). Hence,

$$\varliminf_{\varepsilon\to 0} \varepsilon\sqrt{T_\varepsilon} \ge \tau_0. \tag{2.45}$$

which finishes the first part of the proof of Theorem 1.1. $\square$

§3. Proof of Theorem 1.2

We will use polar coordinates (r, θ, t) instead of (x, t) to study the problem (1.9) and set

$$\sigma = r - t, \quad \tau = \varepsilon\sqrt{t}.$$

Set $u(t,x) = \dfrac{\varepsilon}{\sqrt{r}}\, G(\sigma,\theta,\tau)$ for $r>0$. In this case, it follows from a direct computation that Eq. (1.9) takes the form

$$
\begin{aligned}
P(G) \equiv & -\frac{\varepsilon^2}{\sqrt{rt}}\,\partial^2_{\sigma\tau}G - \frac{\varepsilon^2}{r}\left(c_1\cos^2\theta + c_2\sin^2\theta\right)G\partial^2_\sigma G - \frac{\varepsilon^2}{r}\left(c_1\cos^2\theta + c_2\sin^2\theta\right)(\partial_\sigma G)^2\\
& - \frac{\varepsilon^2}{2r^3}\left((2c_1-c_2)\cos^2\theta + (2c_2-c_1)\sin^2\theta\right)G^2 - \frac{\varepsilon^2}{4t^{3/2}r^{1/2}}\,\partial_\tau G + \frac{\varepsilon^3}{4tr^{1/2}}\,\partial^2_\tau G\\
& - \frac{\varepsilon^2}{r^3}\left(c_1\sin^2\theta + c_2\cos^2\theta\right)(\partial_\theta G)^2 + \frac{2\varepsilon^2}{r^2}\sin\theta\cos\theta(c_1-c_2)(\partial_\theta G)(\partial_\sigma G)\\
& - \frac{\varepsilon}{r^{5/2}}\,\partial^2_\theta G - \frac{\varepsilon}{4r^{5/2}}G - \frac{4\varepsilon^2}{r^3}(c_1-c_2)\sin\theta\cos\theta G\partial_\theta G - \frac{\varepsilon^2}{r^3}\left(c_1\sin^2\theta + c_2\cos^2\theta\right)G\partial^2_\theta G\\
& + \frac{\varepsilon^2}{r^2}\left(2c_1 + 2c_2 - 3c_1\sin^2\theta - 3c_2\cos^2\theta\right)G\partial_\sigma G + \frac{2\varepsilon^2}{r^2}\sin\theta\cos\theta(c_1-c_2)G\partial^2_{\sigma\theta}G = 0.
\end{aligned}
\tag{3.1}
$$

We introduce an unknown transformation Φ by

$$
\Phi(s,\theta,\tau) = (\sigma,\theta,\tau), \tag{3.2}
$$

where $\sigma = \phi(s,\theta,\tau)$, and set

$$
G(\Phi) = v. \tag{3.3}
$$

Therefore, as $\partial_\sigma G = \partial_s v/\partial_s\phi$, if we can find smooth functions ϕ and v satisfying condition (H) and (1.15) of Theorem 1.2, then we will be able to show that the solution u to (1.9) blows up. Under the transformation (3.2) and (3.3), (3.1) takes still another form which is explicitly given in the following lemma:

Lemma 3.1. *Let $R = 1 + \dfrac{\varepsilon^2\phi}{\tau^2}$. Then one has*

$$
-\frac{r}{\varepsilon^2}P(G) \equiv \frac{\partial^2_s\phi\partial_s v}{(\partial_s\phi)^3}\,I_0 + \frac{1}{(\partial_s\phi)^2}\,I_1 + \frac{1}{\partial_s\phi}\,I_2 + I_3 = 0, \tag{3.4}
$$

where

$$
\begin{aligned}
I_0 &= -(c_1\cos^2\theta + c_2\sin^2\theta)v - \frac{\varepsilon^4}{R^2\tau^4}(c_1\sin^2\theta + c_2\cos^2\theta)v(\partial_\theta\phi)^2 - \frac{2\varepsilon^2}{R\tau^2}(c_1-c_2)\sin\theta\cos\theta v\partial_\theta\phi\\
&\quad + R^{1/2}\partial_\tau\phi + \frac{\varepsilon^2 R^{1/2}}{4\tau}(\partial_\tau\phi)^2 - \frac{\varepsilon^2}{R^{3/2}\tau^3}(\partial_\theta\phi)^2,\\
I_1 &= -\partial_s(\partial_s v I_0),\\
I_2 &= Z_1\partial_s v + \varepsilon^2\partial_s v N\phi + \varepsilon^2\partial_s v h_1(\varepsilon,\theta,\tau,v,\partial_\theta v,\phi,\partial_\theta\phi,\partial_\tau\phi),\\
I_3 &= -\varepsilon^2 N v + \varepsilon^2 h_2(\varepsilon,\theta,\tau,v,\partial_\theta v,\partial_\tau v,\phi),
\end{aligned}
$$

h_1, h_2 are smooth functions the explicit expression of which is not required, and the first-order differential operator Z_1 and the second-order differential operator N, respectively, are of the form

$$
\begin{aligned}
Z_1 &= R^{1/2}\big(1 + \frac{\varepsilon^2}{2\tau}\partial_\tau\phi\big)\partial_\tau - \frac{2\varepsilon^2}{R\tau^2}\left(\frac{\varepsilon^2}{R\tau^2}(c_1\sin^2\theta + c_2\cos^2\theta)v\partial_\theta\phi + \sin\theta\cos\theta(c_1-c_2)v + \frac{\partial_\theta\phi}{R^{1/2}\tau}\right)\partial_\theta\\
&\equiv \delta_1\partial_\tau + \varepsilon^2\delta_2\partial_\theta,\\
N &= \frac{R^{1/2}}{4\tau}\partial^2_\tau - \frac{1}{R^{3/2}\tau^3}\left(1 + \frac{\varepsilon^2}{R^{1/2}\tau}(c_1\sin^2\theta + c_2\cos^2\theta)v\right)\partial^2_\theta\\
&\equiv N_1\partial^2_\tau + N_2\partial^2_\theta,
\end{aligned}
$$

where

$$\delta_1 = R^{1/2}\big(1+\frac{\varepsilon^2}{2\tau}\partial_\tau\phi\big), \quad \delta_2 = -\frac{2\varepsilon^2}{R\tau^2}\bigg(\frac{\varepsilon^2}{R\tau^2}(c_1\sin^2\theta + c_2\cos^2\theta)v\partial_\theta\phi + \sin\theta\cos\theta(c_1-c_2)v + \frac{\partial_\theta\phi}{R^{1/2}\tau}\bigg),$$
$$N_1 = \frac{R^{1/2}}{4\tau}, \quad N_2 = -\frac{1}{R^{3/2}\tau^3}\bigg(1+\frac{\varepsilon^2}{R^{1/2}\tau}(c_1\sin^2\theta + c_2\cos^2\theta)v\bigg).$$

It follows from Lemma 3.1 that, in order to solve the nonlinear equation $P(G)=0$, it suffices to solve the system

$$\begin{cases} I_0 = 0, \\ I_2 + \partial_s\phi I_3 = 0, \end{cases} \tag{3.5}$$

which is also called the blowup system for (1.9) in the terminology of [4-5] (where nonlinear wave equations such as $\partial_t^2 v - \Delta_x v + \sum_{0\le i,j,k\le 2} g_{ij}^k \partial_k v \partial_{ij}^2 v = 0$ are dealt with).

The related process is divided into the six parts.

(A) Local existence of a solution to (3.5). ¿From the explicit expression of I_0, one has that $\frac{\partial I_0}{\partial(\partial_\tau\phi)} = R^{1/2} + \frac{\varepsilon^2 R^{1/2}}{2\tau}\partial_\tau\phi > 0$ for $\varepsilon > 0$ small and ϕ a smooth function. By the implicit function theorem, one then obtains from the equation $I_0 = 0$ that

$$\partial_\tau\phi = E(\varepsilon, \theta, \tau, v, \phi, \partial_\theta\phi), \tag{3.6}$$

where E is a smooth function of its arguments.

By §2, for $C_0 > 0$ large enough and $\eta > 0$ sufficiently small, one also has that the equation $P(G) = 0$ can be solved for G in a strip

$$D_S = \big\{(\sigma,\theta,\tau) \mid \sigma\in[-C_0, M],\, \theta\in[\theta^0-\delta_0, \theta^0+\delta_0],\, \tau\in[\tau_1,\tau_1+\eta]\big\}$$

with initial data $\frac{\sqrt{r}}{\varepsilon}u(t,x)$ given at time $t = (\tau_1/\varepsilon)^2$ (since (1.9) has a unique smooth solution there). Here, $\tau_1 > 0$ is a fixed constant satisfying $\tau_1 < \tau_0$, and $\delta_0 > 0$ and $0 < \eta < \tau_0 - \tau_1$ are sufficiently small.

For $\eta > 0$ sufficiently small, Eq. (3.5) then has a unique solution $\overline{\phi}$ with initial data $\overline{\phi}(s,\theta,\tau_1) = s$ (note that the smooth solution $u(t,x)$ of (1.9) exists for $t \le ((\tau_1+\eta)/\varepsilon)^2$, as $G(\sigma,\theta,\tau)$ exists for $\tau \le \tau_1 + \eta$).

Setting $\overline{v} = G(\overline{\phi},\theta,\tau)$ in the strip D_S, one hence gets a local solution to the blowup system (3.5). Moreover, from the uniqueness result on the solution $u(t,x)$ to (1.9) for $t\in\big[0, ((\tau_1+\eta)/\varepsilon)^2\big]$, one has that $\overline{v}$ and $\overline{\phi} - s$ are smooth and flat on $\{s = M\}$.

(B) Choice of the domain and the scalar equation for ϕ. As in [4-5], in order to obtain a weighted energy estimate on the linearized system of (3.5) on a suitable domain D, we choose a "nearly horizontal" surface Σ through $\{\tau=\tau_1, s = M\}$ as part of the boundary of D, where Σ is the characteristic surface of the operator $Z_1\partial_s - \varepsilon^2\partial_s\bar{\phi}N$ the coefficients of which are computed using $(\overline{v},\overline{\phi})$. Let $\tau = \psi(s,\theta)+\tau_1$ be the equation of Σ, where $\psi(M,\theta) = 0$. Then, in view of part (A) and for small $\varepsilon > 0$, $\nabla_{s,\theta}^\alpha\psi = O(\varepsilon^2)$ and $\partial_s\psi \le 0$ holds in D_S for $\alpha\in\mathbb{N}_0^2$.

We choose a cutoff function $\chi\in C^\infty(\mathbb{R})$ with $\chi(p) = 1$ for $p\le\frac{1}{2}$, and $\chi(p) = 0$ for $p \ge 1$ and make the change of variables

$$X = s, \quad Y = \theta, \quad T = \tau - \tau_1 - \psi(s,\theta)\chi\left(\frac{\tau-\tau_1}{\eta}\right). \tag{3.7}$$

The surface Σ then becomes $\{T=0\}$. We will work in the domain $D_1 = \big\{(X,Y,T) \mid -C_0 \le X \le M,$ $\theta^0-\theta_0 \le Y \le \theta^0+\theta_0,\, 0\le T\le \tau_\varepsilon - \tau\big\}$. Note that D_1 is actually unknown at the moment, as we do not know the precise value of τ_ε yet.

Next we derive from (3.5) a scalar equation for ϕ in the new coordinate system (3.7). Since $\partial_v I_0 \neq 0$ for small $\varepsilon > 0$, it follows from $I_0 = 0$ that v can be expressed as

$$v = F(\varepsilon, \theta, \tau, \phi, \partial_\theta \phi, \partial_\tau \phi), \tag{3.8}$$

where F is a smooth function of its arguments. Substituting (3.8) into the second and third equation of (3.5) and going through the direct computations yields

$$L(\phi) \equiv \widetilde{Z_1} SF - \varepsilon^2 (S\phi) \widetilde{N} F + \varepsilon^2 (SF) \widetilde{N} \phi + \varepsilon^2 (SF) \widetilde{h_1} + \varepsilon^2 (S\phi) \widetilde{h_2} = 0, \tag{3.9}$$

where

$$\begin{aligned}
\widetilde{Z_1} &= \widetilde{\delta}_1 \partial_T + \varepsilon^2 \widetilde{\delta}_2 \partial_Y, \quad \widetilde{N} = \widetilde{N}_1 \partial_T^2 + 2\varepsilon^2 \widetilde{N}_2 \partial_{YT}^2 + \widetilde{N}_3 \partial_Y^2, \\
\widetilde{h}_1 &= h_1 - N_1 \psi \chi'' \partial_T \phi / \eta^2 - N_2 \partial_\theta^2 \psi \chi \partial_T \phi, \\
\widetilde{h}_2 &= h_2 + N_1 \psi \chi'' \partial_T F / \eta^2 + N_2 \partial_{\theta_1}^2 \psi \chi \partial_T F, \\
S &= \partial_s = \partial_X - \partial_s \psi \chi \partial_T,
\end{aligned}$$

where

$$\begin{aligned}
&\widetilde{\delta}_1 = \delta_1 \partial_\tau T + \varepsilon^2 \delta_2 \partial_\theta T, \quad \widetilde{\delta}_2 = \delta_2, \\
&\widetilde{N}_1 = N_1 (\partial_\tau T)^2 + N_2 (\partial_\theta T)^2, \quad \widetilde{N}_2 = \varepsilon^{-2} N_2 \partial_\theta T, \quad \widetilde{N}_3 = N_2.
\end{aligned}$$

In order to solve the blowup system (3.5), one hence only needs to solve (3.9) because of (3.8). As in [4-5], we will use the Nash-Moser-Hörmander iteration method to solve Eq. (3.9) under the restriction (H) of Theorem 1.2.

(C) The construction of an approximate solution to (3.9) and the condition (H). As a first step to use the Nash-Moser-Hörmander iteration method, one needs to construct an approximate solution ϕ_a to (3.9) such that ϕ_a satisfies (H) of Theorem 1.2 near some point m_ϵ.

For $\varepsilon = 0$, the blowup system (3.5) becomes

$$(c_1 \cos^2 Y + c_2 \sin^2 Y) v = \partial_T \phi, \quad \partial_T v = 0 \tag{3.10}$$

with the initial value conditions

$$\phi(X, Y, 0) = X, \quad v(X, Y, 0) = F_0(\sigma(X, Y, \tau_1), Y) \tag{3.11}$$

and the boundary condition

$$v|_{X=M} = 0, \tag{3.12}$$

where the function $\sigma(X, Y, \tau_1)$ in (3.11) is determined by $X = \sigma + F_0(\sigma, Y)\tau_1(c_1 \cos^2 Y + c_2 \sin^2 Y)$.

¿From (3.10)-(3.12), one finds a solution to (3.9) for $\varepsilon = 0$, namely

$$\overline{\phi}_0(X, Y, T) = X + T(c_1 \cos^2 Y + c_2 \sin^2 Y) F_0(\sigma(X, Y, \tau_1), Y). \tag{3.13}$$

Note that (3.9) admits a local solution $\overline{\phi}$ for $0 \leq T \leq \eta$ the existence of which has been proven in part (A). Upon glueing $\overline{\phi}$ and $\overline{\phi}_0$ one obtains an approximate solution to (3.9), namely

$$\phi_a(X, Y, T) = \chi\left(\frac{T}{\eta}\right) \overline{\phi}(X, Y, T) + \left(1 - \chi\left(\frac{T}{\eta}\right)\right) \overline{\phi}_0(X, Y, T). \tag{3.14}$$

By a direct verification, one has $L(\phi_a) = f_a$, where f_a is smooth, flat on $\{X = M\}$, and zero near $\{T = 0\}$.

In addition, under the assumption (1.11), one can show that ϕ_a satisfies (H) at the point $(\bar{\sigma}^0, \theta^0, \tau_0 - \tau_1)$ with $\bar{\sigma}^0 = \sigma^0 + (c_1 \sin^2 \theta^0 + c_2 \cos^2 \theta^0) F_0(\sigma^0, \theta^0)\tau_1$:

Lemma 3.2. *The approximate solution ϕ_a constructed in* (3.14) *satisfies* (H) *near the point* $(\bar\sigma^0, \theta^0, \tau_0-\tau_1)$.

Proof. Note that
$$\overline{\phi}_0(X,Y,T) = X + T(c_1\cos^2 Y + c_2\sin^2 Y)F_0(\sigma(X,Y,\tau_1),Y),$$
where $\sigma(X,Y,\tau_1)$ is determined from the expression $X = \sigma + F_0(\sigma,Y)\tau_1(c_1\cos^2 Y + c_2\sin^2 Y)$ (this follows as in the proof of Lemma 2.1).

Set $W(X,Y) = F_0(\sigma(X,Y,\tau_1),Y)(c_1\cos^2\theta + c_2\sin^2\theta)$. First, we assert that
$$\partial_X W(\bar\sigma^0,\theta^0) = \min \partial_X W(X,Y). \tag{3.15}$$
Indeed, it follows from (1.11) and a direct computation that
$$\begin{aligned}
&\nabla_{X,Y}\partial_X W(\bar\sigma^0,\theta^0) = 0,\\
&\nabla_X^2\partial_X W(\bar\sigma^0,\theta^0) = \frac{(c_1\cos^2\theta^0 + c_2\sin^2\theta^0)\partial_\sigma^3 F_0(\sigma^0,\theta^0)}{(1+\tau_1(c_1\cos^2\theta^0 + c_2\sin^2\theta^0)\partial_\sigma F_0(\sigma^0,\theta^0))^4},\\
&\nabla_{XY}^2\partial_X W(\bar\sigma^0,\theta^0) = \frac{-\tau_1\partial_\theta((c_1\cos^2\theta + c_2\sin^2\theta)F_0)(\sigma^0,\theta^0)}{(1+\tau_1(c_1\cos^2\theta^0 + c_2\sin^2\theta^0)\partial_\sigma F_0(\sigma^0,\theta^0))^4}\partial_\sigma^3 F_0(\sigma^0,\theta^0)(c_1\cos^2\theta^0 + c_2\sin^2\theta^0)\\
&\qquad + \frac{\partial_\theta\big(\partial_\sigma^2 F_0(c_1\cos^2\theta + c_2\sin^2\theta)\big)(\sigma^0,\theta^0)}{(1+\tau_1(c_1\cos^2\theta^0 + c_2\sin^2\theta^0)\partial_\sigma F_0(\sigma^0,\theta^0))^3},
\end{aligned}$$
and
$$\begin{aligned}
\nabla_Y^2\partial_X W(\bar\sigma^0,\theta^0) = {}& \frac{\tau_1^2\big(\partial_\theta((c_1\cos^2\theta + c_2\sin^2\theta)F_0)(\sigma^0,\theta^0)\big)^2}{(1+\tau_1(c_1\cos^2\theta^0 + c_2\sin^2\theta^0)\partial_\sigma F_0(\sigma^0,\theta^0))^4}(c_1\cos^2\theta^0 + c_2\sin^2\theta^0)\partial_\sigma^3 F_0(\sigma^0,\theta^0)\\
&- \frac{2\tau_1\partial_\theta((c_1\cos^2\theta + c_2\sin^2\theta)F_0)(\sigma^0,\theta^0)}{(1+\tau_1(c_1\cos^2\theta^0 + c_2\sin^2\theta^0)\partial_\sigma F_0(\sigma^0,\theta^0))^3}\partial_\theta((c_1\cos^2\theta + c_2\sin^2\theta)\partial_\sigma^2 F_0)(\sigma^0,\theta^0)\\
&+ \frac{\partial_\theta^2((c_1\cos^2\theta + c_2\sin^2\theta)\partial_\sigma F_0)(\sigma^0,\theta^0)}{(1+\tau_1(c_1\cos^2\theta^0 + c_2\sin^2\theta^0)\partial_\sigma F_0(\sigma^0,\theta^0))^2}.
\end{aligned}$$
This, together with $\nabla_{\sigma,\theta}^2[\partial_\sigma F_0(\sigma,\theta)(c_1\cos^2\theta + c_2\sin^2\theta)]|_{(\sigma,\theta)=(\sigma^0,\theta^0)} > 0$, yields by a direct, but tedious computation
$$\nabla_{X,Y}^2\partial_X W(\bar\sigma^0,\theta^0) > 0. \tag{3.16}$$
Thus, the assertion (3.15) has been shown. Moreover, by the uniqueness of the minimum point of the function $\partial_\sigma F_0(\sigma,\theta)(c_1\cos^2\theta + c_2\sin^2\theta)$, one has that $(\bar\sigma^0,\theta^0,\tau_1)$ is also the unique minimum point of $\partial_X W(X,Y)$.

We now establish that ϕ_a satisfies (H) near the point $(\bar\sigma^0,\theta^0,\tau_0-\tau_1)$.

(i) By $\partial_X\overline{\phi}(X,Y,0) = 1$ and the smallness of $\eta > 0$, one can assume that, for $T \le \eta$,
$$\partial_X\overline{\phi}(X,Y,T) > 0.$$
In addition,
$$\partial_X\overline{\phi}_0(X,Y,T) = 1 + T\partial_X W(X,Y). \tag{3.17}$$
If $\partial_X W(X,Y) \ge 0$ at some point (X,Y,τ_1), then $\partial_X\overline{\phi}_0(X,Y,T) \ge 1$. If $\partial_X W(X,Y) < 0$ at some point (X,Y,τ_1), then due to the fact $T \le T_0 = -\min\dfrac{1}{\partial_X\big(F_0(\sigma(X,Y,\tau_1))\big)\big(c_1\cos^2 Y + c_2\sin^2 Y\big)} = \tau_0 - \tau_1$ one has from (3.17) that
$$\partial_X\overline{\phi}_0(X,Y,T) \ge 1 + \partial_X W(X,Y)T_0 \ge 0.$$

Consequently, $\partial_X \phi_a(X, Y, T) \geq 0$ holds.

On the other hand, $\partial_X \phi_a(X, Y, T) = 0$ holds if and only if $T \geq \eta$ and $\partial_X \overline{\phi}_0(X, Y, T) = 0$ which gives

$$\partial_X \phi_a(X, Y, T) = 0 \iff (X, Y, T) = (\bar{\sigma}^0, \theta^0, \tau_0 - \tau_1).$$

(ii) It follows from the expression for ϕ_a and the smallness of $\eta > 0$ that in the neighborhood of $(\bar{\sigma}^0, \theta^0, \tau_0 - \tau_1)$

$$\phi_a(X, Y, T) = X + TW(X, Y) \tag{3.18}$$

which gives $\partial^2_{XT} \phi_a(\bar{\sigma}^0, \theta^0, \tau_0 - \tau_1) < 0$. In addition, in view of $\nabla_{X,Y} \partial_X W(X, Y)(\bar{\sigma}^0, \theta^0) = 0$, (3.16), and (3.18), one readily obtains

$$\nabla_{X,Y} \partial_X \phi_a(\bar{\sigma}^0, \theta^0, \tau_0 - \tau_1) = 0, \qquad \nabla^2_{X,Y} \partial_X \phi_a(\bar{\sigma}^0, \theta^0, \tau_0 - \tau_1) > 0.$$

Collecting all the assertions above concludes the proof of Lemma 3.2. $\square$

(D) Goursat problem for the nonlinear equation (3.9) on a fixed domain. In order to adjust the height of the domain D_1 as in [4] we perform a change of variables depending on a parameter λ close to zero,

$$X = x, \quad Y = y, \quad T = T(\rho, \lambda) = (\tau_0 - \tau_1)(\rho + \lambda\rho(1 - \chi_1(\rho))), \tag{3.19}$$

where χ_1 is 1 near 0 and 0 near 1. ¿From now on we will be working on a fixed subdomain of D_1,

$$D_2 = \left\{(x, y, \rho) \mid -C_0 \leq x \leq M, \, \theta^0 - \delta_0 \leq y \leq \theta^0 + \delta_0, \, 0 \leq \rho \leq 1\right\}$$

and write Eq. (3.9) as

$$L(\lambda, \phi) = 0. \tag{3.20}$$

For $\lambda = \lambda_0 = 0$, the approximate solution to (3.20) is

$$\phi_0(x, y, \rho) = \phi_a(x, y, T(\rho, 0)) = \phi_a(x, y, (\tau_0 - \tau_1)\rho),$$

where $L(\lambda_0, \phi_0) = f_0(x, y, \rho) = f_a(x, y, (\tau_0 - \tau_1)\rho)$. Moreover, ϕ_0 satisfies (H) in D_2 at some point $(x_0, y_0, 1)$ by part (C).

On the characteristic surfaces $\{x = M\}$ and $\{\rho = 0\}$ of Eq. (3.20), we impose the natural boundary conditions

$$\phi \text{ is flat on } \{x = M\} \text{ and } \phi - \phi_0 \text{ is flat on } \{\rho = 0\}, \text{ respectively.} \tag{3.21}$$

(E) Linearizing (3.20) under the condition (H). In order to solve (3.20) together with (3.21) in the domain D_2 under the condition (H), we are required to linearize (3.20) suitably.

Denote the linearized operator of L by

$$L'(\lambda, \phi)(\dot{\lambda}, \dot{\phi}) = \partial_\lambda L(\lambda, \phi)\dot{\lambda} + \partial_\phi L(\lambda, \phi)\dot{\phi} = \dot{f}. \tag{3.22}$$

In addition, if $L(\lambda, \phi) = f$, then taking the derivative with respect to the variable λ yields

$$\partial_\lambda L(\lambda, \phi) + \partial_\phi L(\lambda, \phi)\left(\partial_\rho \phi \frac{\partial_\lambda T}{\partial_\rho T}\right) = \partial_\rho f \frac{\partial_\lambda T}{\partial_\rho T}. \tag{3.23}$$

Therefore, if one wants to solve $L(\lambda, \phi) = f$ for a small right-hand side f, then it follows from the standard Nash-Moser-Hörmander iteration method that we are only required to solve the linearized equation $L'(\lambda, \phi)(\dot{\lambda}, \dot{\phi}) = \dot{f}$ and provide the needed tame estimate (see [3]). From (3.22)-(3.23), one has

$L'(\lambda,\phi)(\dot\lambda,\dot\phi) = \partial_\phi L(\lambda,\phi)\Big(\dot\phi - \dot\lambda\partial_\rho\phi\dfrac{\partial_\lambda T}{\partial_\rho T}\Big) + \dot\lambda\partial_\rho f\dfrac{\partial_\lambda T}{\partial_\rho T}$. Setting $\dot\Phi = \dot\phi - \dot\lambda\partial_\rho\phi\dfrac{\partial_\lambda T}{\partial_\rho T}$, it suffices to solve the equation

$$\begin{cases} \partial_\phi L(\lambda,\phi)\dot\Phi = \dot f, \\ \dot\Phi \text{ is flat on both } \{x=M\} \text{ and } \{\rho=0\} \end{cases} \tag{3.24}$$

for a right-hand side $\dot f$ which is also flat on both $\{x=M\}$ and $\{\rho=0\}$, since the second-order error term (here $\dot\lambda\partial_\rho f\dfrac{\partial_\lambda T}{\partial_\rho T}$) does not play an essential role in the Nash-Moser-Hörmander iteration (see [3]).

It follows from a direct, but tedious computation concerning $\partial_\phi L_i(\lambda,\phi)$ that from (3.24) one obtains

$$\begin{cases} \overline{P}\dot\Phi \equiv ZSZ\dot\Phi - \varepsilon^2(S\phi)QZ\dot\Phi + \varepsilon^2 l(\dot\Phi) = \dot f, \\ \dot\Phi \text{ is flat on both } \{x=M\} \text{ and } \{\rho=0\} \end{cases} \tag{3.25}$$

as the linearized problem of (3.20), where

$$Z = \partial_\rho + \varepsilon^2 z_0\partial_y, \quad S = \partial_x + \varepsilon^2 s_0\partial_\rho, \quad Q = Q_1 Z^2 + 2\varepsilon^2 Q_2 Z\partial_y + Q_3\partial_y^2,$$

here z_0, s_0, and Q_i are smooth. More specifically,

$$z_0 = z_0(x,\rho,\lambda,\phi,\partial_y\phi,\partial_\rho\phi), \quad s_0 = s_0(x,y,\rho,\lambda),$$
$$Q_1 = \frac{1}{4(\tau_1+T)\partial_\rho T} + O(\varepsilon^2), \quad Q_3 = -\frac{\partial_\rho T}{(\tau_1+T)^3} + O(\varepsilon^2),$$

and l is a second-order operator which is a linear combination of $id, S, Z, \partial_y, SZ, Z^2, Z\partial_y, \partial_y^2$ and whose coefficients depend on the derivatives of ϕ up to third order.

(F) The tame estimate and solvability of (3.24). Comparing the operator $\bar P$ in (3.25) with the operator $\partial_\phi\mathcal{L}(\lambda,\phi)$ in Proposition IV.1 of [4], one sees that $\bar P$ is just of the form of $\partial_\phi\mathcal{L}(\lambda,\phi)$ with $B\equiv 0$ and $b_0\equiv 0$. By carefully checking the proofs of Proposition IV.2.2, Proposition IV.3.1, and Proposition IV.4 of [4], one then has under the condition (H) on the function ϕ near some point $(\bar x_0,\bar y,1)$:

Lemma 3.3. *There exists a subdomain D_0 of D_2 which is a domain of influence domain for the first-order differential operator $\tilde Z_1$ in* (3.9)*, that contains the point $(\bar x_0,\bar y,1)$, and that is bounded by the planes $\{x=-C_0\}$, $\{x=M\}$, $\{\rho=0\}$, $\{\rho=1\}$ with the followoing property: If $|\phi-\phi_0|_{C^7(D_3)}\le\varepsilon_0$ with ε_0 a small positive constant and if $f\in C^\infty(D_3)$ is flat on both $\{x=M\}$ and $\{\rho=0\}$, then* (3.25) *has a unique smooth solution in D_3. Moreover, one has the energy estimate* (*i.e., tame estimate*)

$$|\dot\Phi|_s \le C_s\left(|\dot f|_{s+n_0} + |\dot f|_{n_0}(1+|\phi|_{s+n_0})\right), \tag{3.26}$$

for any $s\in\mathbb{N}$, where $|\cdot|_s = \|\cdot\|_{H^s(D_3)}$ and $n_0\in\mathbb{N}$ is some fixed integer.

Based on Lemma 3.3 and the standard Nash-Moser-Hörmander iteration method (see [3, 4-5]), and using $\partial_X v(\bar\sigma^0,\theta^0)\ne 0$ in (3.11) and the Sobolev imbedding theorem, we have now completed the proof of Theorem 1.2.

§4. Proof of Theorem 1.1

Using Theorem 1.2, we now conclude the proof of Theorem 1.1.

Recall that so far we have obtained the C^3 solution ϕ to Eq. (3.20), (3.21) in the domain D_0. By (3.8), we immediately obtain v in D_0. Indeed, we have solved the modified blowup system (3.5) in D_0. Therefore, the solution to (1.9) is obtained in the domain $\Phi(D_0)$ in the coordinate system (s,θ,τ), hence

$\|u\|_{C(\Phi(D_0))} \le C\varepsilon^2$. Now we go back to the original coordinate system (r,θ,t) so that the conclusions of Theorem 1.1 can be obtained.

For $(s,\theta,\tau) \in D_0$ close to the point m_ε given in Theorem 1.2, by Taylor's formula and condition (H) in Theorem 1.2, there exists a point $(\bar{s},\bar{\theta},\bar{\tau}) = (\bar{\lambda}s + (1-\bar{\lambda})s_\varepsilon, \bar{\lambda}\theta + (1-\bar{\lambda})\theta_\varepsilon, \bar{\lambda}\tau + (1-\bar{\lambda})\tau_\varepsilon)$ with $0 < \bar{\lambda} < 1$ such that

$$\begin{aligned} \partial_s\phi(s,\theta,\tau) &= \partial^2_{s\tau}\phi(m_\varepsilon)(\tau-\tau_\varepsilon) \\ &\qquad + \frac{1}{2}(s-s_\varepsilon,\theta-\theta_\varepsilon,\tau-\tau_\varepsilon)\nabla^2_{s,\theta,\tau}\partial_s\phi(\bar{s},\bar{\theta},\bar{\tau})(s-s_\varepsilon,\theta-\theta_\varepsilon,\tau-\tau_\varepsilon)^T. \end{aligned} \tag{4.1}$$

In addition, we may assume $-2c_0 \le \partial^2_{s\tau}\phi \le -c_0$ in D_0 since $\partial^2_{s\tau}\phi(m_\varepsilon) < 0$ and $\phi \in C^3(D_0)$, here $c_0 > 0$ is a constant. Together with $\nabla^2_{s,\theta}\partial_s\phi(m_\varepsilon) > 0$, for $(s,\theta,\tau) \in D_0$, this yields

$$\partial_s\phi(s,\theta,\tau) \ge c_0(\tau_\varepsilon - \tau) = c_0\varepsilon\,\frac{T_\varepsilon - t}{\sqrt{T_\varepsilon}+\sqrt{t}} \ge \frac{c_0\varepsilon}{4}\cdot\frac{T_\varepsilon - t}{\sqrt{t}}. \tag{4.2}$$

Furthermore, if $|(s-s_\varepsilon,\theta-\theta_\varepsilon)| < \tau_\varepsilon - \tau$, then

$$|\partial_s\phi(s,\theta,\tau)| \le 3c_0(\tau_\varepsilon-\tau) \le c_0\varepsilon\,\frac{T_\varepsilon - t}{\sqrt{t}}. \tag{4.3}$$

¿From the expression for $u = \dfrac{\varepsilon}{\sqrt{r}}\,v$, one has

$$\begin{aligned} \partial_1 u &= -\frac{\varepsilon\cos\theta}{2r^{3/2}}\,v - \frac{\varepsilon}{\sqrt{r}}\left(\partial_\theta v - \frac{\partial_s v}{\partial_s\phi}\,\partial_\theta\phi\right)\frac{\sin\theta}{r}, \\ \partial_2 u &= -\frac{\varepsilon\sin\theta}{2r^{3/2}}\,v + \frac{\varepsilon}{\sqrt{r}}\left(\partial_\theta v - \frac{\partial_s v}{\partial_s\phi}\,\partial_\theta\phi\right)\frac{\cos\theta}{r}, \\ \partial_t u &= \frac{\varepsilon^2}{2\sqrt{rt}}\left(\partial_\tau v - \frac{\partial_s v}{\partial_s\phi}\,\partial_\tau\phi\right). \end{aligned}$$

Substituting (4.2)-(4.3) into the these formulas yields

$$\frac{1}{C\,(T_\varepsilon - t)} \le \|\partial_t u\|_{L^\infty(\Phi(D_0)} \ \text{ and } \ \|\nabla_{t,x}u(t,\cdot)\|_{L^\infty(\Phi(D_0))} \le \frac{C}{T_\varepsilon - t}. \tag{4.4}$$

Owing to assumption (1.11), outside $\Phi(D_0)$ and for $t \le T_\varepsilon$, the smooth solution of (2.1) does not blow up in $(\{t \le T_\varepsilon\}\times\mathbb{R}^3)\setminus\Phi(D_0)$. Therefore, similar to the proof of Lemma 2.4, one obtains in $(\{t \le T_\varepsilon\}\times\mathbb{R}^3)\setminus\Phi(D_0)$ that

$$|\partial u| \le C\varepsilon\,(1+t)^{-1/2} \text{ and } |u| \le C\varepsilon.$$

Finally, by Theorem 1.2 and the related Nash-Moser-Hörmander iteration process, one concludes that $\lim\limits_{\varepsilon\to 0}\tau_\varepsilon = \tau_0$ for the solution $u(t,x)$ when the variables $(r-t,\theta,\varepsilon\sqrt{t})$ lie in $\Phi(D_0)$. This implies that the lifespan T_ε satisfies

$$\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\sqrt{T}_\varepsilon \le \tau_0. \tag{4.5}$$

Together with (2.45), this yields

$$\lim_{\varepsilon\to 0}\varepsilon\sqrt{T}_\varepsilon = \tau_0,$$

which completes the proof of Theorem 1.1.

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Department of Mathematics and IMS, Nanjing University, Nanjing 210093, P.R. China

E-mail address: balancenjust@yahoo.com.cn

Mathematical Institute, University of Göttingen, Bunsenstr. 3-5, D-37073 Göttingen, Germany
E-mail address: iwitt@uni-math.gwdg.de

Department of Mathematics and IMS, Nanjing University, Nanjing 210093, P.R. China
E-mail address: huicheng@nju.edu.cn